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Explicit Formulae for Some Kazhdan–Lusztig R-Polynomials

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We consider the Kazhdan–Lusztig R-polynomials, $R_{u,v}(q)$ indexed by permutations "u, v" having particular forms. More precisely, we show that $R_{e,34\cdots n12}(q)$ (where "e" denotes the identity permutation) equals, aside from a simple change of variable, a q-analogue of the Fibonacci number, and if two permutations are obtained one from the other by applying two transpositions (one simple, and one not), then the corresponding R-polynomial factors nicely. Our proofs are combinatorial.

1. INTRODUCTION

The *R-polynomials* are a family of polynomials, defined for every Coxeter group W, which are intimately related to the multiplicative structure of the Hecke algebra associated to W (see [8, Sect. 2]). These polynomials were introduced by Kazhdan and Lusztig with the intent of proving the existence of another family of polynomials; in fact in their fundamental paper, [8], they defined another family of polynomials for every Coxeter group W. These polynomials are indexed by pairs of elements of W and are known as the Kazhdan–Lusztig polynomials of W (see, e.g., [7, Chap. 7]). They are intimately related to the Bruhat order of W and to the algebraic geometry of Schubert varieties; moreover they are of fundamental importance in representation theory. In this sense, the importance of the R-polynomials of W (see, e.g., [7, Sect. 7.5]) stems mainly from the fact that their knowledge is equivalent to that of the Kazhdan–Lusztig polynomials.

In recent years purely combinatorial rules to compute the R-polynomials have been found, whereas they are usually defined algebraically in terms of Hecke algebras (see, e.g., [7]). These rules not only make these objects

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more concrete, but also allow combinatorial reasoning and techniques to be applied to them.

In this paper we particularly refer to the work done in this sense by Brenti on the R-polynomials for the symmetric groups (see [2]). More precisely, in [2] he points out a connection between the R-polynomials and the enumeration and combinatorics of increasing subsequences in permutations; with this aim he defined a new family of polynomials basically related to these increasing subsequences. A delicate combinatorial analysis brings the author to show that these polynomials are substantially the R-polynomials of the symmetric groups (see [2, Corollary 3.8]).

In this theory fundamental problems are to characterize these polynomials and to find closed formulas for different classes of permutations.

Our aim in this paper is to show a relation between R-polynomials and Fibonacci polynomials and a closed product formula for R-polynomials which are indexed by a pair of permutations (u, v), where v is obtained from u by swapping four elements.

The organization of the paper is the following.

In the next section we recall some basic definitions, notation, and results, both of an algebraic and combinatorial nature that will be used afterwards. In the third section we define R-polynomials and $\widetilde{R}$-polynomials and recall their properties. In Section 4 we show a relationship between R-polynomials and Fibonacci polynomials. In the last one we prove our main result, a closed formula for the R-polynomials for a certain class of permutations.

2. NOTATION AND PRELIMINARIES

In this section we collect some definitions, notation, and results that will be used in the rest of this paper.

We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, ...\}$, $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$, and $\mathbf{Z}$ be the set of integers; for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, 3, ..., a\}$ (where $[0] \stackrel{\text{def}}{=} \varnothing$).

Given $n, m \in \mathbf{P}$, $n \leqslant m$, we let $[n, m] = [m] \backslash [n-1]$. We write $S = \{a_1, ..., a_r\}_<$ to mean that $S = \{a_1, ..., a_r\}$ and $a_1 < \cdots < a_r$. The cardinality of a set A will be denoted with $|A|$.

Given a set T we will let $S(T)$ be the set of all bijections of T in itself and $S_n \stackrel{\text{def}}{=} S([n])$.

If $\sigma \in S(T)$ and $T \stackrel{\text{def}}{=} \{t_1, ..., t_n\}_< \subseteq \mathbf{P}$ then we write $\sigma = \sigma_1 \cdots \sigma_n$ to mean that $\sigma(t_i) = \sigma_i$, for $i = 1, ..., n$. If $\sigma \in S_n$ then we will also write σ on *disjoint cycle form* (see, e.g., [10, p. 17]) and we will not usually write the 1-cycles of σ. For example, if $\sigma = 365492187$ then $\sigma = (1, 3, 5, 9, 7)(2, 6)$.

Given $\sigma, \tau \in S_n$ then $\sigma\tau = \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(1, 3) = (1, 3, 2)$. We will follow [7] for general Coxeter group notation and terminology. Given a Coxeter system (W, S) and $\sigma \in W$ we denote

$$D(\sigma) \stackrel{\text{def}}{=} \{s \in S : \ell(\sigma s) < \ell(\sigma)\};$$

$D(\sigma)$ is called the *descent set of* σ. We denote by e the identity of W, and we let $T \stackrel{\text{def}}{=} \{wsw^{-1} : s \in S, w \in W\}$, which is called the *reflection set* of W.

We will always assume that W is partially ordered by (strong) *Bruhat order*. We recall (see, e.g., [7, Sect. 5.9]) that this means that if $u, v \in W$, $u \leqslant v$ iff $\exists t_1, ..., t_r \in T$, for $r \in \mathbf{N}$ such that:

(i) $v = ut_1 t_2 \cdots t_r$

(ii) $\ell(ut_1 \cdots t_{i+1}) = \ell(ut_1 \cdots t_i)$ for $i = 0, ..., r-1$.

The polynomials $R_{x, w}(q)$ defined by the next theorem are called the *R-polynomials* of W:

Theorem 2.1. *There is a unique family of polynomials* $\{R_{x, w}(q)\}_{x, w \in W} \subseteq \mathbf{Z}[q]$ *such that*:

$$\begin{aligned}
&\text{(i)} \quad R_{x, w}(q) = 0, \qquad \textit{if} \quad x \nleqslant w;\\
&\text{(ii)} \quad R_{x, w}(q) = 1, \qquad \textit{if} \quad x = w;\\
&\text{(iii)} \quad R_{x, w}(q) = \begin{cases} R_{xs, ws}(q), & \textit{if} \quad s \in D(x) \\ (q-1)\, R_{x, ws}(q) + qR_{xs, ws}(q), & \textit{if} \quad s \notin D(x) \end{cases},\\
&\qquad\qquad \textit{if} \quad x < w \quad \textit{and} \quad s \in D(w).
\end{aligned} \tag{1}$$

See [7, Sect. 7.5] for a proof.

It is important to point out that the above theorem can be used to give an inductive procedure to compute the R-polynomials of W because $\ell(ws) < \ell(w)$.

From now on we assume $W = S_n$ and $S = \{s_1, ..., s_{n-1}\}$, where $s_i \stackrel{\text{def}}{=} (i, i+1)$, for $i \in [n-1]$. It is useful to have combinatorial descriptions of the three objects, which are important to R-polynomials computation. So now we review the Bruhat order for the symmetric group with a well known characterization.

For $u \in S_n$, and $i \in [n]$, let $\{u^{i,1}, ..., u^{i,i}\}_< \stackrel{\text{def}}{=} \{u(1), ..., u(i)\}$.

THEOREM 2.2. *Let $u, v \in S_n$. Then $u \leqslant v$ iff $u^{i,j} \leqslant v^{i,j}$ for every $1 \leqslant j \leqslant i \leqslant n-1$.*

A proof of this result can be found in [6].

For example: if $u = 15432$ and $v = 24531$ then $(u^{1,1}, u^{2,1}, u^{2,2}, u^{3,1}, u^{3,2}, u^{3,3}, u^{4,1}, u^{4,2}, u^{4,3}, u^{4,4}) = (1, 1, 5, 1, 4, 5, 1, 3, 4, 5)$ and $(v^{1,1}, v^{2,1}, v^{2,2}, v^{3,1}, v^{3,2}, v^{3,3}, v^{4,1}, v^{4,2}, v^{4,3}, v^{4,4}) = (2, 2, 4, 2, 4, 5, 2, 3, 4, 5)$, so u and v are incomparable.

Finally, in the following proposition, we give characterizations for the length function and descent set in the symmetric group. We refer the reader to [9] for a proof.

PROPOSITION 2.3. *Let $w \in S_n$, and $i \in [n-1]$. Then*

(i) $\ell(w) = \mathrm{inv}(w) \stackrel{\text{def}}{=} |\{(i, j) \in [n] \times [n] : i < j, w(i) > w(j)\}|$; *the number* $\mathrm{inv}(w)$ *is usually known as inversions of* w.

(ii) $s_i \in D(u)$ *iff* $u(i) > u(i+1)$.

For example, if $u = 15432$ then $\mathrm{inv}(u) = |\{(2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}| = 6$ and $D(u) = \{(2, 3), (3, 4), (4, 5)\}$.

In the rest of the paper a descent $(i, i+1)$ may be written briefly as i.

3. THE R-POLYNOMIALS OF THE SYMMETRIC GROUP

In this section we will follow [2], describing a combinatorial rule for the R-polynomial for S_n, based on the enumeration and combinatorics of increasing subsequences of a permutation.

To introduce this rule we need the next definitions.

For $u \in S_n$ and $i, j \in [n]$, $i \neq j$, we define $C_{i,j}(u) \stackrel{\text{def}}{=} \{(u(i_1), u(i_2), ..., u(i_k)) \in S_n : k \in [n], i = i_1 < i_2 < \cdots < i_k = j \text{ and } u(i_1) < u(i_2) < \cdots < u(i_k)\}$; this is the set of *increasing subsequences of the permutation u from i to j*.

For example, we can see that if $u = 215496378$, then $C_{1,6}(u) = \{(2, 6), (2, 5, 6), (2, 4, 6)\}$. Note that in general $C_{i,j}(u)$ is not empty iff $i < j$ and $u(i) < u(j)$.

Now we are going to define a distance-function on S_n that will have a crucial role in the rest of this paper. For the properties of this function we refer the reader to [2].

For $u, v \in S_n$ let $d(u, v) \stackrel{\text{def}}{=} \max\{i \in [n] : u^{-1}(i) \neq v^{-1}(i)\}$, where $\max\{\varnothing\} \stackrel{\text{def}}{=} 0$. For example, $d(198265374, 298461357) = \max\{1, 2, 5, 7, 4\} = 7$.

Now we can define a new class of polynomials in terms of increasing subsequences. For $u, v \in S_n$, we define a polynomial $\tilde{R}_{u,v}(t)$ with the next algorithm,

$$\widetilde{R}_{u,v}(t) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if} \quad u \not\leqslant v \\ \displaystyle\sum_{w \in C_{v^{-1}(d),\, u^{-1}(d)}(u)} t^{k(w)-1} \widetilde{R}_{wu,v}(t), & \text{if} \quad u < v \\ 1, & \text{if} \quad u = v, \end{cases} \tag{2}$$

where $d \stackrel{\text{def}}{=} d(u, v)$ and $k(w)$ is the length of the longest cycle of w.

Before going on we will illustrate this definition with an example. Let $u = 2147563$ and $v = 6157243$. By Theorem 2.2 $u < v$. It results that $d = 6$, $C_{1,6}(u) = \{(2,6), (2,4,6), (2,5,6), (2,4,5,6)\}$; hence by definition (2) $\widetilde{R}_{2147563,\, 6157243}(t) = t\widetilde{R}_{6147523,\, 6157243}(t) + t^2 \widetilde{R}_{6147253,\, 6157243}(t) + t^2 \widetilde{R}_{6127543,\, 6157243}(t) + t^3 \widetilde{R}_{6127453,\, 6157243}(t)$.

In order to show that the polynomials $\widetilde{R}_{u,v}(t)$ defined in (2) "are the R-polynomials of the symmetric group" we are going to recall their following fundamental property:

THEOREM 3.1. *Let $u, v \in S_n$ such that $u \leqslant v$. Then, for every $s \in D(v)$, we have that*

$$\widetilde{R}_{u,v}(t) = \begin{cases} \widetilde{R}_{us,vs}(t), & \textit{if} \quad s \in D(u), \\ \widetilde{R}_{us,vs}(t) + t\widetilde{R}_{u,vs}(t), & \textit{if} \quad s \notin D(u). \end{cases} \tag{3}$$

The next result states the precise relationship between the polynomial (2) and the R-polynomial of S_n.

COROLLARY 3.2. *Let $u, v \in S_n$; then*

$$R_{u,v}(q) = q^{(\ell(v) - \ell(u))/2} \widetilde{R}_{u,v}(q^{1/2} - q^{-1/2}).$$

This is the fundamental result in [2, Corollary 3.8]. We note that one of the advantages of working with the polynomials $\widetilde{R}_{u,v}(t)$ is that they have positive integer coefficients while the R-polynomials have integer coefficients, but after Corollary 3.2 we can see that every result on the $\widetilde{R}_{u,v}(t)$ can be translated into a result on R-polynomials.

Note that Theorem 3.1 can be used as an inductive procedure to compute $\widetilde{R}_{u,v}(t)$ since $\mathrm{inv}(v(i, i+1)) = \mathrm{inv}(v) - 1$.

There is one more general fact on the $\widetilde{R}_{u,v}(t)$ which we will use:

PROPOSITION 3.3. *Let $u, v \in S_n$; then*

$$\begin{aligned} \widetilde{R}_{u,v}(t) &= \widetilde{R}_{u^{-1},v^{-1}}(t) = \widetilde{R}_{n+1-v(1)\cdots n+1-v(n),\, n+1-u(1)\cdots n+1-u(n)}(t) \\ &= \widetilde{R}_{v(n)\cdots v(1),\, u(n)\cdots u(1)}(t). \end{aligned}$$

The above result can be proved easily using properties of Hecke algebra and Corollary 3.2 [see [7, Proposition 7.6]].

As we said before, our aim in this paper is to prove some explicit formulas for the $\tilde{R}_{u,v}(t)$-polynomials. We note that a general closed formula for them does not exist; for example,

$$\tilde{R}_{12345,\,54321}(t) = t^2(1+5t^2+10t^4+6t^6+t^8)$$

and

$$\tilde{R}_{123456,\,654321}(t) = t^3(1+9t^2+39t^4+57t^6+36t^8+10t^{10}+t^{12}),$$

and these factors are irreducible over the field of rational numbers.

However, there are several general classes of permutations for which explicit simple formulas exist. We refer the reader to [3] for a survey of the main results known in this direction.

4. A RELATIONSHIP BETWEEN R-POLYNOMIALS AND FIBONACCI POLYNOMIALS

Definition. Let $n \in \mathbf{N}$. Then we will call the polynomial $F(n) \stackrel{\text{def}}{=} F(n-1) + tF(n-2)$, where $F(0) \stackrel{\text{def}}{=} 1$ and $F(1) \stackrel{\text{def}}{=} 1$, the nth Fibonacci polynomial. (See [4].)

Theorem 4.1. *Let $n>1$. Then*

$$\tilde{R}_{12\cdots n,\,34\cdots n12}(t) = t^{2(n-2)}F(n-2)_{|t^{-2}}$$

where $F(n-2)_{|t^{-2}}$ is the $(n-2)$th Fibonacci-polynomial in t^{-2}.

Proof. We proceed by induction on n, the thesis being clearly true if $n=3$. So let $n \geqslant 4$. To prove the theorem we will show first that

$$\begin{aligned}\tilde{R}_{12\cdots(n-1)(n-2)\,n,\,34\cdots n12}(t) = t^2(&\tilde{R}_{12\cdots n,\,34\cdots(n-1)\,12n}(t)\\ &+\tilde{R}_{12\cdots n,\,34\cdots(n-2)\,12(n-1)\,n}(t)).\end{aligned} \tag{4}$$

If $\sigma = 12\cdots n$ and $\tau = 34\cdots n12$ then $D(\sigma)=\varnothing$ and $D(\tau)=\{(n-2)\}$; therefore by 3.1 it follows that

$$\begin{aligned}\tilde{R}_{12\cdots(n-1)(n-2)\,n,\,34\cdots n12}(t) = {}&\tilde{R}_{12\cdots(n-1)(n-2)\,n,\,34\cdots(n-1)\,1n2}(t)\\ &+t\tilde{R}_{12\cdots n,\,34\cdots 1n2}(t).\end{aligned} \tag{5}$$

We calculate the first summand on the right hand side of (5).

Note that $n-3 \in D(34\cdots(n-1)\,1n2)$ but $n-3 \notin D(12\cdots(n-1)(n-2)\,n)$; so by 3.1

$$\tilde{R}_{12\cdots(n-1)(n-2)\,n,\,34\cdots(n-1)\,1n2}(t) = \tilde{R}_{12\cdots(n-1)(n-3)(n-2)\,n,\,34\cdots 1(n-1)\,n2}(t) + t\tilde{R}_{12\cdots(n-1)(n-2)\,n,\,34\cdots 1(n-1)\,n2}(t).$$

By 2.2, $(12\cdots(n-1)(n-3)(n-2)\,n) \nleqslant (34\cdots 1(n-1)\,n2)$; thus by definition (2) the first summand of the last equality vanishes. Applying 3.1 before to the descent $n-1$ in $(34\cdots 1(n-1)\,n)$ and after to the descent $n-2$ in $(34\cdots 1(n-1)\,2n)$, we have

$$\begin{aligned} t\tilde{R}_{12\cdots(n-1)(n-2)\,n,\,34\cdots 1(n-1)\,n2}(t) &= t(\tilde{R}_{12\cdots(n-1)\,n(n-2),\,34\cdots 1(n-1)\,2n}(t) \\ &\quad + t\tilde{R}_{12\cdots(n-1)(n-2)\,n,\,34\cdots 1(n-1)\,2n}(t)) \\ &= 0 + t^2\tilde{R}_{12\cdots(n-2)(n-1)\,n,\,34\cdots 12(n-1)\,n}(t). \end{aligned} \tag{6}$$

Now we are going to evaluate the second summand on the right hand side of (5),

$$\begin{aligned} t\tilde{R}_{12\cdots n,\,34\cdots 1n2}(t) &= t\tilde{R}_{12\cdots n(n-1),\,34\cdots 12n}(t) + t^2\tilde{R}_{12\cdots n,\,34\cdots(n-1)\,12n}(t) \\ &= t^2\tilde{R}_{12\cdots n,\,34\cdots(n-1)\,12n}(t), \end{aligned} \tag{7}$$

by Theorem 2.2 and Definition (2). So from (6) and (7) follows (4).

By Proposition 3.3,

$$\tilde{R}_{12\cdots n,\,34\cdots(n-1)\,12n}(t) = \tilde{R}_{12\cdots(n-1),\,34\cdots(n-1)\,12}(t) \qquad \text{and}$$

$$\tilde{R}_{12\cdots n,\,34\cdots(n-2)\,12(n-1)\,n}(t) = \tilde{R}_{12\cdots(n-2),\,34\cdots(n-2)\,12}(t).$$

Finally from (4) and the inductive hypothesis,

$$\begin{aligned} \tilde{R}_{12\cdots n+1,\,34\cdots(n+1)\,12}(t) &= t^2\tilde{R}_{12\cdots n,\,34\cdots n12}(t) + t^2\tilde{R}_{12\cdots(n-1),\,34\cdots(n-1)\,12}(t) \\ &= t^2(t^{(2n-4)}F(n-2)_{|t^{-2}}) + t^2(t^{(2n-6)}F(n-3)_{|t^{-2}}) \\ &= t^{(2n-2)}(F(n-2)_{|t^{-2}} + t^{-2}F(n-3)_{|t^{-2}}) \\ &= t^{(2n-2)}F(n-1)_{|t^{-2}}, \end{aligned}$$

since by definition $F(n)_{|t^{-2}} = F(n-1)_{|t^{-2}} + t^{-2}F(n-2)_{|t^{-2}}$. ∎

Corollary 4.2. $R_{id,\,34\cdots n12}(t) = (t+1)^{2(n-1)}F(n-2)_{|(t-2+t^{-1})^{-1}}$.

Proof. This follows easily by application of Theorem 4.1 and Corollary 3.2. ∎

5. THE MAIN RESULT

In this section we derive our main result; this gives a closed formula for the R-polynomial of a pair of permutations (u, v), where v is obtained from u by swapping four elements in a certain way, namely $v = u(i, j)(k, k+1)$, where $i < k < k+1 < j$.

This formula will be the object of Theorem 5.4; before that we will give two preliminary lemmas, which will be used for the proof. The first one is an analysis of the Bruhat order relation between u and v under our particular hypothesis, whose verification we leave to the reader, the second one is a reduction lemma.

LEMMA 5.1. *Let $u \in S_n$, $k \in D(u)$, $1 \leqslant i < k < k+1 < j \leqslant n$ and suppose that $v = u(i, j)(k, k+1)$. Then $u \leqslant v$ iff $u(i) < u(k+1) < u(k) < u(j)$.*

LEMMA 5.2. *Let $u \in S_n$, $k \in D(u)$, $1 < k < k+1 < n$ and suppose that $v = u(1, n)(k, k+1)$. If $D(u) \cap D(v) = \varnothing$, $u(1) = 1$ and $u(n) = n$, then $\widetilde{R}_{u,v}(t) = \widetilde{R}_{(k,k+1),(1,n)}(t)$.*

Proof. By the assumptions, we have $u = 1u(2)\, u(3) \cdots u(k-2)\, u(k-1)\, u(k)\, (k+1)\, u(k+2) \cdots n$, with $u(k) > u(k+1)$, and $v = nu(2)\, u(3) \cdots u(k-2)\; u(k-1)\; u(k+1)\; u(k)\, u(k+2) \cdots 1$.

First we observe that since $D(u) \cap D(v) = \varnothing$, it is clearly true that the set $D(v)$ contains at least the descents 1 and $n-1$ and at most the descent $(k-1)$ and $(k+1)$.

To start we see the obvious case that $D(v) = \{1, n-1\}$.

It follows immediately that the next inequality chain holds:

$$1 < u(2) < u(3) < u(4) < \cdots < u(k-2) < u(k-1) < u(k+1)$$
$$< u(k) < \cdots < u(n-1) < n.$$

Then $u = 123 \cdots (k-1)(k+1)\, k(k+2) \cdots (n-1)\, n = (k, k+1)$ and $v = n234 \cdots (k-1)\, k(k+1) \cdots (n-1)\, 1 = (1, n)$ so the thesis.

Now suppose that $D(v) = \{1, (n-1), (k-1)\}$, by hypothesis it must be $u(k) > u(k-1)$ because if it is not then $(k-1) \in D(u) \cap D(v)$ which is a contradiction.

So we have $1 < u(2) < u(3) < \cdots < u(k-2) < u(k-1) < u(k) < u(k+2) < \cdots < u(n-1) < n$ and $u(k+1) < u(k-1)$, and we can say that $u(j) = j$ for every $j \in \{k+2, \ldots, n-1\}$.

Let $p \stackrel{\text{def}}{=} \max\{m \in \mathbf{N} : 2 \leqslant m \leqslant k-2, u(m) < u(k+1)\}$ where $p \stackrel{\text{def}}{=} 1$ if the set is empty, then

$$u(2) < \cdots < u(p-1) < u(p) < u(k+1) < u(p+1) < \cdots < u(k-3)$$
$$< u(k-2) < u(k-1) < u(k) < k+2$$

so that $u = 123 \cdots (p-1)\, p(p+2) \cdots (k-2)(k-1)\, k(k+1)(p+1)(k+2) \cdots n$, $v = n23 \cdots (p-1)\, p(p+2) \cdots (k-2)(k-1)\, k(p+1)(k+1)(k+2) \cdots 1$. Therefore

$$u^{-1} = 123 \cdots (p-1)\, p(k+1)(p+1)(p+2)$$
$$\cdots (k-4)(k-3)(k-2)(k-1)\, k(k+2) \cdots n$$
$$v^{-1} = n23 \cdots (p-1)\, pk(p+1)(p+2)$$
$$\cdots (k-4)(k-3)(k-2)(k-1)(k+1)(k+2) \cdots (n-1)\, 1.$$

At this point, applying Proposition 3.3, we have $\tilde{R}_{u,v}(t) = \tilde{R}_{u^{-1}, v^{-1}}(t)$ and then by Theorem 3.1, considering that $(p+1) \in D(v^{-1}) \cap D(u^{-1})$, $\tilde{R}_{u^{-1}, v^{-1}}(t) = \tilde{R}_{u_1, v_1}(t)$, where

$$u_1 = 123 \cdots (p-1)\, p(p+1)(k+1)(p+2)$$
$$\cdots (k-4)(k-3)(k-2)(k-1)\, k(k+2) \cdots n$$

and

$$v_1 = n23 \cdots (p-1)\, p(p+1)(k)(p+2)$$
$$\cdots (k-4)(k-3)(k-2)(k-1)\, k+1(k+2) \cdots 1.$$

Now $(p+2) \in D(v_1) \cap D(u_1)$, and we apply 3.1 again to the descent $(p+2)$.

Iterating the same procedure we arrive at the situation in which $k+1$ is in the kth position in the first permutation and k is in the same position in the second one. So we will have a polynomial equal to the one at the beginning but calculated on the pair of permutations $(k, k+1)$, $(1, n)$.

Suppose now that $D(v) = \{1, (n-1), (k+1)\}$, i.e., $u(k) > u(k+2)$. Let $s \stackrel{\text{def}}{=} \max\{m \in \mathbf{N} : k+1 < m < n-1, u(k) > u(m)\}$; then we can say that $u(k+1) < u(k+2) < \cdots < u(s-1) < u(s) < u(k) < u(s+1) < \cdots < u(n-1) < u(n)$, because of the hypothesis that u and v have no common descents.

This forces that

$$v = n23\cdots(k-2)(k-1)\,ks(k+1)\cdots(s-2)(s-1)(s+1)\cdots(n-1)\,1$$

and

$$u = 123\cdots(k-2)(k-1)\,sk(k+1)\cdots(s-2)(s-1)(s+1)\cdots(n-1)\,n.$$

We calculate the inverse permutations of u and v,

$$u^{-1} = 123\cdots(k-2)(k-1)(k+1)(k+2)\cdots(s-1)\,sk(s+1)\cdots n$$

$$v^{-1} = n23\cdots(k-2)(k-1)\,k(k+2)\cdots(s-1)\,s(k+1)(s+1)\cdots 1.$$

Then by Proposition 3.3, $\tilde{R}_{u,v}(t) = \tilde{R}_{u^{-1},v^{-1}}(t)$.

By Theorem 3.1 applied to the common descent $(s-1)$ we have that $\tilde{R}_{u^{-1},v^{-1}}(t) = \tilde{R}_{u_1,v_1}(t)$, where $u_1 = 123\cdots(k-1)(k+1)\cdots(s-1)\,ks(s+1)\cdots(n-1)\,n$ and $v_1 = n23\cdots(k-1)\,k\cdots(s-1)(k+1)\,s(s+1)\cdots(n-1)\,1$.

Now observe that $(s-2) \in D(v_1) \cap D(u_1)$, so by Theorem 3.1 we have $\tilde{R}_{u_1,v_1}(t) = \tilde{R}_{u_2,v_2}(t)$, where

$$u_2 = 123\cdots(k-1)(k+1)\cdots(s-2)\,k(s-1)\,s(s+1)\cdots(n-1)\,n$$

and

$$v_2 = n23\cdots(k-1)(k+1)\cdots(s-2)(k+1)(s-1)\,s(s+1)\cdots(n-1)\,1.$$

Iterating this procedure one obtains a polynomial which is equal to the one at the beginning but calculated on the permutations $(k, k+1)$ and $(1, n)$, as we wanted to prove.

Finally we investigate the case in which $D(v) = \{1, (k-1), (k+1), (n-1)\}$. This forces

$$u(k-1) > u(k+1) \quad \text{but} \quad u(k-1) < u(k) \quad \text{and}$$

$$u(k) > u(k+2) \quad \text{but} \quad u(k+1) < u(k+2).$$

Let p and s be defined as before; we have to consider two different situations:

(i) $u(k-1) < u(k+2)$;

(ii) $u(k+2) < u(k-1)$.

(iii) If $u(k-1)<u(k+2)$ then by the hypothesis

$$\begin{aligned}1&<u(2)<\cdots<u(p)<u(k+1)<u(p+1)<\cdots\\&<u(k-1)<u(k+2)<u(k+3)<\cdots<u(s-1)\\&<u(s)<u(k)<u(s+1)<\cdots<u(n-1)<n.\end{aligned}$$

Therefore it must be the case that $u=12\cdots p(p+2)\cdots ks(p+1)(k+1)\cdots(s-1)(s+1)\cdots(n-1)\,n$ and $v=n2\cdots p(p+2)\cdots k(p+1)\,s(k+1)\cdots(s-1)(s+1)\cdots(n-1)\,1$.

We calculate the inverse of u and v:

$$\begin{aligned}u^{-1}&=1\cdots p(k+1)(p+1)\cdots(k-2)(k-1)(k+2)(k+3)\\&\quad\cdots sk(s+1)\cdots(n-1)\,n\\v^{-1}&=n\cdots pk(p+1)\cdots(k-2)(k-1)(k+2)(k+3)\cdots s(k+1)(s+1)\\&\quad\cdots(n-1)\,1.\end{aligned}$$

By Proposition 3.3 we have $\tilde{R}_{u,v}(t)=\tilde{R}_{u^{-1},v^{-1}}(t)$.

We can observe that $\{(p+1),(s-1)\}\in D(u^{-1})\cap D(v^{-1})$ and by repeated application of 3.1 we obtain the thesis as in the previous cases.

(ii) Suppose that $u(k+2)<u(k-1)$ then $u(k+1)<u(k+2)<u(k-1)<u(k)$. We define

$$a\overset{\text{def}}{=}\max\{j\in[n]:p\leqslant j\leqslant k-2\text{ and }u(j)<u(k+2)\};$$

$$b\overset{\text{def}}{=}\max\{j\in[n]:k+2\leqslant j\leqslant s\text{ and }u(j)<u(k-1)\}.$$

Then we have the next inequality chain:

$$\begin{aligned}1&<u(2)<\cdots<u(p)<u(k+1)<u(p+1)<\cdots\\&<u(a)<u(k+2)<u(a+1)\\&<\cdots<u(k-2)<\cdots<u(b)<u(k-1)<u(b+1)<\cdots\\&<u(s-1)<u(s)<u(k)<u(s+1)<\cdots<u(n-1)<n.\end{aligned}$$

It follows that $u(i)=i$ for every $i\in[p]$ and for every i such that $s+1\leqslant i\leqslant n-1$,

$$
\begin{array}{lll}
u(i)=i+1 & \text{for every } i \text{ such that} & p+1 \leqslant i \leqslant a, \\
u(i)=i+2 & \text{for every } i \text{ such that} & a+1 \leqslant i \leqslant k-2;
\end{array}
$$

$$u(k-1)=(b-1), \qquad u(k)=s, \qquad u(k+1)=(p+1), \qquad u(k+2)=(a+2);$$

$$
\begin{array}{lll}
u(i)=i-2 & \text{for every } i \text{ such that} & k+3 \leqslant i \leqslant b \quad \text{and} \\
u(i)=i-1 & \text{for every } i \text{ such that} & b+1 \leqslant i \leqslant s.
\end{array}
$$

Now we have

$$
\begin{aligned}
u^{-1} &= 1 \cdots p(k+1)(p+1) \cdots (a-1)\, a(k+2) \cdots (k-3)(k-2)(k+3)(k+4) \\
&\quad \cdots b(k-1)(b+1) \cdots sk(s+1) \cdots (n-1)\, n \\
v^{-1} &= n \cdots pk(p+1) \cdots (a-1)\, a(k+2) \cdots (k-3)(k-2)(k+3)(k+4) \\
&\quad \cdots b(k-1)(b+1) \cdots s(k+1)(s+1) \cdots (n-1)\, 1.
\end{aligned}
$$

By Proposition 3.3, $\tilde{R}_{u,v}(t) = \tilde{R}_{u^{-1},v^{-1}}(t)$.

Now the result follows by repeated application of Theorem 3.1, as in the previous case. ∎

This lemma will enable us to reduce the proof of the Theorem 5.4 to compute $\tilde{R}_{(k,k+1),(1,n)}(t)$; before going on with the last result, we need to point out another preliminary observation. It is an easy application of Theorem 3.1 and Proposition 3.3 that will be very useful for the induction process on which substantially depends the proof of Theorem 5.4.

Proposition 5.3. $\tilde{R}_{(k,k+1),(1,n)}(t) = (1+t^2)\, \tilde{R}_{(n-k,n-k+1),(1,n-1)}(t)$.

Proof. We apply Theorem 3.1 to the descent $1 \in D((1,n))$, obtaining $\tilde{R}_{(k,k+1),(1,n)}(t) = \tilde{R}_{21\cdots(k+1)k\cdots(n-1)n,2n34\cdots(n-1)1}(t) + t\tilde{R}_{(k+1,k),2n34\cdots(n-1)1}(t)$.

We compute $\tilde{R}_{21\cdots(k+1)k\cdots(n-1)n,2n34\cdots(n-1)1}(t)$: observe that the inverse permutation of $u_1 = 21\cdots(k+1)\,k\cdots(n-1)\,n$ is itself, while the inverse of $v_1 = 2n34\cdots(n-1)\,1$ is $v_2 = n134\cdots(n-1)\,2$, so by Proposition 3.3 we have $\tilde{R}_{21\cdots(k+1)k\cdots(n-1)n,2n34\cdots(n-1)1}(t) = \tilde{R}_{u_1,n134\cdots(n-1)2}(t)$. Now if $u \in S_n$, we can define the complementary permutation $\bar{u}$ of u as follows $\bar{u} = n+1-u(1)\cdots n+1-u(n)$, so by Proposition 3.3,

$$
\begin{aligned}
\tilde{R}_{u_1,v_2}(t) &= \tilde{R}_{1n(n-2)\cdots 32(n-1),(n-1)n(n-2)\cdots(n-k)(n-k+1)\cdots 21}(t) \\
&= \tilde{R}_{12\cdots(n-k+1)(n-k)\cdots(n-2)n(n-1),(n-1)2\cdots(n-k)(n-k+1)\cdots(n-2)n1}(t),
\end{aligned}
$$

and finally, by Theorem 3.1, we have $\tilde{R}_{21\cdots(k+1)k\cdots(n-1)n,2n\cdots k(k+1)\cdots(n-1)1}(t) = \tilde{R}_{(n-k,n-k+1),(1,n-1)}(t)$.

We compute now $\tilde{R}_{(k+1,k),2n\cdots k(k+1)\cdots(n-1)1}(t)$: by Proposition 3.3 we have $\tilde{R}_{(k+1,k),2n\cdots k(k+1)\cdots(n-1)1}(t) = \tilde{R}_{(k,k+1),n13\cdots(n-1)2}(t)$ and then by

Theorem 3.1, $\tilde{R}_{(k,k+1),n13\cdots(n-1)2}(t)=\tilde{R}_{21\cdots(k+1)k\cdots(n-1)n,1n3\cdots(n-1)2}(t)+t\tilde{R}_{(k,k+1),1n3\cdots(n-1)2}(t)=t\tilde{R}_{(k,k+1),1n3\cdots(n-1)2}(t)$, since the first summand vanishes by Theorem 2.2 and Definition (2). Now applying as before Proposition 3.4 we have

$$\begin{aligned}\tilde{R}_{(k,k+1),1n3\cdots(n-1)2}(t)&=\tilde{R}_{n1(n-2)\cdots2(n-1),n(n-1)\cdots(n-k)(n-k+1)\cdots1}(t)\\&=\tilde{R}_{(n-k,n-k+1),(1,n-1)}(t).\end{aligned}$$

So $\tilde{R}_{(k,k+1),(1,n)}(t)=\tilde{R}_{(n-k,n-k+1),(1,n-1)}(t)+t^2\tilde{R}_{(n-k,n-k+1),(1,n-1)}(t)$. ∎

THEOREM 5.4. *Let* $u\in S_n$, $k\in D(u)$, $1\leqslant i<k<k+1<j\leqslant n$ *and suppose that* $v=u(i,j)(k,k+1)$ *and* $u<v$. *Then* $\tilde{R}_{u,v}(t)=t^4(1+t^2)^{(\mathrm{inv}(v)-\mathrm{inv}(u)-4)/2}$.

Proof. We can assume that $i=1$, $j=n$ and $u(1)=1$ $u(n)=n$ (this follows from Lemma 5.1 and Proposition 3.3).

We proceed by induction on $d=\max\{i\in[n]:u^{-1}(i)\neq v^{-1}(i)\}$, which is trivially true if $d=4$ (observe that by definition of v and Lemma 5.1 we have $d=n$).

It follows from Theorem 3.1 that we can suppose $D(u)\cap D(v)=\emptyset$, so by Lemma 5.2 we have to compute only $\tilde{R}_{(k,k+1),(1,n)}(t)$.

By Proposition 5.3 we know that $\tilde{R}_{(k,k+1),(1,n)}(t)=(1+t^2)\tilde{R}_{(n-k,n-k+1),(1,n-1)}(t)$ and since $d((n-k+1,n-k),(1,n-1))=n-1$ we can apply the induction hypothesis to obtain

$$\tilde{R}_{(n-k,n-k+1),(1,n-1)}(t)=t^4(1+t^2)^{(\mathrm{inv}(v)-\mathrm{inv}(u)-2-4)/2}$$

(together with the fact that $\mathrm{inv}(1,n-1)-\mathrm{inv}(n-k+1,n-k)=\mathrm{inv}(v)-\mathrm{inv}(u)-2$) and the thesis follows easily. ∎

COROLLARY 5.5. *Under the hypothesis of Theorem* 5.4,

$$R_{u,v}(t)=(t-1)^4\,(1-t+t^2)^{(\mathrm{inv}(v)-\mathrm{inv}(u)-4)/2}.$$

Proof. This follows immediately from Corollary 3.2 and Theorem 5.4. ∎

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